LIMITING DISTRIBUTIONS OF CURVES UNDER GEODESIC FLOW ON HYPERBOLIC MANIFOLDS

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Abstract. We consider the evolution of a compact segment of an analytic curve on the unit tangent bundle of a finite volume hyperbolic n-manifold under the geodesic flow. Suppose that the curve is not contained in a stable leaf of the flow. It is shown that under the geodesic flow, the normalized parameter measure on the curve gets asymptotically equidistributed with respect to the normalized natural Riemannian measure on the unit tangent bundle of a closed totally geodesically immersed submanifold.

Moreover, if this immersed submanifold is a proper subset, then a lift of the curve to the universal covering space $\mathrm{T}^1(\mathbb{H}^n)$ is mapped into a proper subsphere of the ideal boundary sphere $\partial\mathbb{H}^n$ under the visual map. This proper subsphere can be realized as the ideal boundary of an isometrically embedded hyperbolic subspace in $\mathbb{H}^n$ covering the closed immersed submanifold.

In particular, if the visual map does not send a lift of the curve into a proper subsphere of $\partial\mathbb{H}^n$, then under the geodesic flow the curve gets asymptotically equidistributed on the unit tangent bundle of the manifold with respect to the normalized natural Riemannian measure.

The proof uses dynamical properties of unipotent flows on finite volume homogeneous spaces of $\mathrm{SO}(n,1)$.

1. Introduction

It is instructive to note the following dynamical property: Let $\psi : I = [0,1] \to \mathbb{R}^n$ be a C^2-curve such that for any proper rational hyperplane, say $\mathcal{H}$, in $\mathbb{R}^n$, the set $\{s \in I : \psi(s) \in \mathcal{H}\}$ has null measure. Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, and let $\pi : \mathbb{R}^n \to \mathbb{T}^n$ denote the quotient map. Then for any continuous function f on $\mathbb{T}^n$,

$$\lim_{\alpha\to\infty} \int_0^1 f(\pi(\alpha\psi(s)))\,\mathrm{d}s = \int_{\mathbb{T}^n} f(x)\,\mathrm{d}x, \tag{1}$$

where $\mathrm{d}x$ denotes the normalized Haar integral on $\mathbb{T}^n$. Using Fourier transforms, we can verify (1) for the characters

$$f_m(x) := \exp(2\pi(m\cdot x)), \quad \forall x \in \mathbb{T}^n, \text{ where } m \in \mathbb{Z}^n.$$

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The above observation was used in [8] for $\psi(t) = (\cos(2\pi t), \sin(2\pi t))$; the unit circle in $\mathbb{R}^2$. Later we learnt that a general result in this direction was obtained earlier by B. Randol [10] in response to a question raised by D. Sullivan.

Now we ask a similar question for the hyperbolic spaces. Consider the unit ball model B^n for the hyperbolic n-space $\mathbb{H}^n$ of constant curvature (-1). Let $\Gamma \subset \mathrm{SO}(n,1)$ be a discrete subgroup such that $M := \mathbb{H}^n/\Gamma$ is a hyperbolic manifold of finite Riemannian volume. Let $\pi : \mathbb{H}^n \to M$ be the quotient map. As a special case of a more general result proved in [3, 4], we have that if we project the invariant probability measure on the sphere $\alpha\mathbb{S}^{n-1} \subset B^n$, for $0 < \alpha < 1$, under π to M, then asymptotically as $\alpha \to 1^-$, the measure gets equidistributed with respect to the normalized measure associated to the Riemannian volume form on M. The case of $n = 3$ was proved earlier in [10].

In this article, we will address the following much more refined problem: Instead of the invariant measure on the sphere, we take a smooth measure on a one-dimensional curve on $\mathbb{S}^{n-1}$ and describe the limiting distribution of the projection of its expands on $\alpha\mathbb{S}^{n-1}$ as $\alpha \to 1^-$.

Theorem 1.1. *Let $\bar{\psi} : I = [0,1] \to \mathbb{S}^{n-1}$ be an analytic map. If $\bar{\psi}(I)$ is not contained in a proper subsphere in $\mathbb{S}^{n-1}$, then for any $f \in \mathrm{C_c}(M)$,*

$$\lim_{\alpha \to 1^-} \int_I f(\pi(\alpha\bar{\psi}(s))\,\mathrm{d}s = \int_M f(x)\,\mathrm{d}x, \tag{2}$$

where $\mathrm{d}x$ denotes the normalized integral associated to the Riemannian volume form on M.

By a *proper subsphere* of $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ we mean the intersection of $\mathbb{S}^{n-1}$ with a proper affine subspace of $\mathbb{R}^n$.

Now we describe a generalization of the above phenomenon in a suitable geometric framework. Let $\partial\mathbb{H}^n$ denote the ideal boundary of $\mathbb{H}^n$. Let $\mathrm{T}^1(\mathbb{H}^n)$ denote the unit tangent bundle on $\mathbb{H}^n$. We identify $\partial\mathbb{H}^n$ with $\mathbb{S}^{n-1}$. Let

$$\mathrm{Vis} : \mathrm{T}^1(\mathbb{H}^n) \to \partial\mathbb{H}^n \cong \mathbb{S}^{n-1},$$

denote the visual map sending a tangent to the equivalence class of the directed geodesics tangent to it. Thus any fiber of the visual map is a (weakly) stable leaf of the geodesic flow. Now let M be any n-dimensional hyperbolic manifold (with constant curvature (-1) and) with finite Riemannian volume, let $\mathrm{T}^1(M)$ denote the unit tangent bundle on M, and let $\{g_t\}$ denote the geodesic flow on $\mathrm{T}^1(M)$. Let $\pi : \mathbb{H}^n \to M$ be a universal covering map, and let $\mathrm{D}\,\pi : \mathrm{T}^1(\mathbb{H}^n) \to \mathrm{T}^1(M)$ denote its derivative.

Theorem 1.2. *Let $\psi : I = [a,b] \to \mathrm{T}^1(M)$ be an analytic curve such that $\mathrm{Vis}(\tilde{\psi}(I))$ is not a singleton set, where $\tilde{\psi} : I \to \mathrm{T}^1(\mathbb{H}^n)$ denotes a lift of ψ to the covering space; that is, $\mathrm{D}\,\pi \circ \tilde{\psi} = \psi$. Then there exists a totally geodesic immersion $\Phi : M_1 \to M$ of a hyperbolic manifold M_1 with finite volume such that the following holds: $\forall f \in \mathrm{C_c}(\mathrm{T}^1(M))$,*

$$\lim_{t\to\infty} \frac{1}{|I|} \int_I f(g_t\psi(s))\,\mathrm{d}s = \int_{\mathrm{T}^1(M_1)} f((\mathrm{D}\,\Phi)(v))\,\mathrm{d}v, \tag{3}$$

where $|\cdot|$ denotes the Lebesgue measure, and $\mathrm{d}v$ denotes the normalized integral on $\mathrm{T}^1(M_1)$ associated to the Riemannian volume form on M_1.

Moreover if $\pi' : \mathbb{H}^m \to M_1$ denotes a locally isometric covering map, then there exists an isometric embedding $\tilde{\Phi} : \mathbb{H}^m \hookrightarrow \mathbb{H}^n$ such that

$$\pi \circ \tilde{\Phi} = \Phi \circ \pi' \ \textit{and}\ \mathrm{Vis}(\tilde{\psi}(I)) \subset \partial(\tilde{\Phi}(\mathbb{H}^m)).$$

In order to describe the relation between $\mathrm{Vis}(\tilde{\psi}(I))$ and the totally geodesic immersion Φ, we will recall the following:

Theorem 1.3 ([12],[13]). *Let M be a hyperbolic manifold with finite Riemannian volume. For $k \geq 2$, let $\Psi : \mathbb{H}^k \to M$ be a totally geodesic immersion. Then there exists a totally geodesic immersion $\Phi : M_1 \to M$ of a hyperbolic manifold M_1 with finite Riemannian volume such that*

$$\overline{\Psi(\mathbb{H}^k)} = \Phi(M_1) \quad \textit{and} \quad \overline{\mathrm{D}\,\Psi(\mathrm{T}^1(\mathbb{H}^k))} = \mathrm{D}\,\Phi(\mathrm{T}^1(M_1)).$$

□

This result can be obtained as a direct consequence of the orbit closure theorem for unipotent flows (Raghunathan's conjecture) proved by Ratner [12]; more specifically, the fact that the closure of any $\mathrm{SO}(k,1)$-orbit in $\mathrm{SO}(n,1)/\Gamma$ is a closed orbit of a subgroup of the form $Z \cdot \mathrm{SO}(m,1)$, where Z is a compact subgroup of the centralizer of $\mathrm{SO}(m,1)$ in $\mathrm{SO}(n,1)$.

Remark 1.1. Let the notation be as in Theorem 1.2. Let $\mathbb{S}^{k-1}$ be the smallest dimensional subsphere of $\partial\mathbb{H}^n \cong \mathbb{S}^{n-1}$ such that $\mathrm{Vis}(\tilde{\psi}(I)) \subset \mathbb{S}^{k-1}$. Since $\mathrm{Vis}(\tilde{\psi}(I))$ is not a singleton set, we have $2 \leq k \leq n$. Therefore there exists an isometric embedding $\mathbb{H}^k \hookrightarrow \mathbb{H}^n$ such that $\partial\mathbb{H}^k = \mathbb{S}^{k-1}$. If $\{\tilde{g}_t\}$ denotes the geodesic flow and $\tilde{d}(\cdot,\cdot)$ denotes the distance function on $\mathrm{T}^1(\mathbb{H}^n)$, then

$$\lim_{t\to\infty} \sup_{s\in I} \tilde{d}(\tilde{g}_t\tilde{\psi}(s), \mathrm{T}^1(\mathbb{H}^k)) = 0. \tag{4}$$

Since $\pi : \mathbb{H}^k \to M$ is a totally geodesic immersion, by Theorem 1.3 there exists a totally geodesic immersion $\Phi : M_1 \to M$ of a hyperbolic

manifold of finite Riemannian volume such that

$$\Phi(M_1) = \overline{\pi(\mathbb{H}^k)}. \tag{5}$$

This describes the map Φ as involved in the statement of Theorem 1.2. Also, by (4) and (5), if $d(\cdot,\cdot)$ denotes the distance function on M then

$$\lim_{t\to\infty} \sup_{s\in I} d(g_t\psi(s), \mathrm{D}\,\Phi(\mathrm{T}^1(M_1))) = 0. \tag{6}$$

We have the following consequences.

Theorem 1.4. *Let M be a hyperbolic Riemannian manifold with finite volume. Let $\psi : I \to M$ be an analytic map such that $\mathrm{Vis}(\tilde{\psi}(I))$ is not contained in a proper subsphere in $\partial\mathbb{H}^n$, where $\tilde{\psi} : I \to \mathrm{T}^1(\mathbb{H}^n)$ is a lift of ψ such that $\mathrm{D}\,\pi\circ\tilde{\psi} = \psi$. Then given any $f \in \mathrm{C_c}(\mathrm{T}^1(M))$,*

$$\lim_{t\to\infty} \frac{1}{|I|}\int_I f(a_t\psi(s))\,\mathrm{d}s = \int_{\mathrm{T}^1(M)} f\,\mathrm{d}v,$$

where $\mathrm{d}v$ is the normalized integral on $\mathrm{T}^1(M)$ associated to the Riemannian volume form on M.

Corollary 1.5. *Let M be a hyperbolic manifold with finite volume. Let $x \in M$ and $\psi : I = [a,b] \to {\mathrm{T}^1}_x(M)$ be an analytic map such that $\psi(I)$ is not contained in any proper subsphere in ${\mathrm{T}^1}_x(M)$. Then*

$$\lim_{t\to\infty} \frac{1}{|I|}\int_I f(g_t\psi(s))\,\mathrm{d}t = \int_{\mathrm{T}^1(M)} f(v)\,\mathrm{d}v, \quad \forall f \in \mathrm{C_c}(\mathrm{T}^1(M)),$$

where $\mathrm{d}v$ is the normalized Riemannian volume integral on $\mathrm{T}^1(M)$.

It may be interesting to compare the above result with [16] where any rectifiable invariant set for the geodesic flow is shown to be a conull subset of the unit tangent bundle of a closed finite volume totally geodesic submanifold.

1.1. **Reformulation in terms of flows on homogeneous spaces.**

Let $G = \mathrm{SO}(n,1)$, and P^- be a minimal parabolic subgroup of G, and $K \cong \mathrm{SO}(n)$ be a maximal compact subgroup of G. Then $M := P^- \cap K \cong \mathrm{SO}(n-1)$. Since $G = P^-K$,

$$P^-\backslash G \cong M\backslash K \cong \mathrm{SO}(n-1)\backslash SO(n) \cong \mathbb{S}^{n-1}. \tag{7}$$

We let $p : G \to \mathbb{S}^{n-1}$ denote the quotient map corresponding to (7). Let A be a maximal connected $\mathbb{R}$-diagonalizable subgroup of G contained in $\mathrm{Z_G}(M) \cap P^-$. Since G is of $\mathbb{R}$-rank 1, A is a one-parameter group,

and the centralizer of A in G is $\mathrm{Z_G}(A) := MA$. Let N^- denote the unipotent radical of P^-. Define

$$\begin{aligned} (8)\ A^+ &= \{a \in A : a^k g a^{-k} \to e \text{ as } k \to \infty \text{ for any } g \in N^-\}, \text{ and} \\ (9)\ \ N &= \{g \in G : a^{-k} g a^k \to e \text{ as } k \to \infty \text{ for any } a \in A^+\}. \end{aligned}$$

Let $\mathfrak{n}$ denote the Lie algebra on N. Then $\mathfrak{n}$ is abelian, and we identify it with $\mathbb{R}^{n-1}$. Let $u : \mathbb{R}^{n-1} \to N$ be the map $u(v) = \exp(v)$ for any $v \in \mathbb{R}^{n-1} \cong \mathfrak{n}$. We observe that the map

$$S : \mathbb{R}^{n-1} \to \mathbb{S}^{n-1} \text{ defined by } S(v) = p(u(v)), \quad \forall v \in \mathbb{R}^{n-1}, \tag{10}$$

is the inverse of stereographic projection.

Let $\alpha : A \to \mathbb{R}^*$ be the character such that $au(v)a^{-1} = u(\alpha(a)v)$ for all $v \in \mathbb{R}^{n-1}$. Then $A^+ = \{a \in A : \alpha(a) > 1\}$.

Let Γ be a lattice in G and μ_G be the G-invariant probability measure on G/Γ.

Theorem 1.6. *Let $\theta : I = [a,b] \to G$ be an analytic map such that $p(\theta(I))$ is not contained in a subsphere of $\mathbb{S}^{n-1}$. Then given any $f \in \mathrm{C_c}(G/\Gamma)$, any compact set $\mathcal{K} \subset G/\Gamma$ and any $\epsilon > 0$, there exists $R > 0$ such that for any $a \in A^+$ with $\alpha(a) > R$,*

$$\left| \frac{1}{|I|} \int_I f(a\theta(t)x)\,\mathrm{d}t - \int_{G/\Gamma} f\,\mathrm{d}\mu_G \right| < \epsilon, \quad \forall x \in \mathcal{K}. \tag{11}$$

First we shall consider the following crucial case of the above theorem.

Theorem 1.7. *Let $\varphi : I = [a,b] \to \mathbb{R}^{n-1}$ be an analytic curve such that $\varphi(I)$ is not contained in any sphere or an affine hyperplane. Let $x_i \xrightarrow{i\to\infty} x$ be a convergent sequence in G/Γ, and let $\{a_i\}_{i\in\mathbb{N}}$ be a sequence in A^+ such that $\alpha(a_i) \xrightarrow{i\to\infty} \infty$. Then*

$$\lim_{i\to\infty} \frac{1}{|I|} \int_I f(a_i u(\varphi(t)x_i)\,\mathrm{d}t = \int_{G/\Gamma} f\,\mathrm{d}\mu_G, \quad \forall f \in \mathrm{C_c}(G/\Gamma). \tag{12}$$

We will deduce the above result from the following general statement, which is the main result of this paper.

Theorem 1.8. *Let $\varphi : I \to \mathbb{R}^{n-1}$ be a nonconstant analytic map, and let $x \in G/\Gamma$. Then there exist a closed subgroup H of G, an analytic map $\zeta : I \to M (= \mathrm{Z_G}(A) \cap K)$ and $h_1 \in G$ such that $\pi(H)$ is closed and admits a finite H-invariant measure, say μ_H, and the following holds: For any sequence $\{a_i\}_{i\in\mathbb{N}} \subset A^+$, if $\alpha(a_i) \xrightarrow{i\to\infty} \infty$ then*

$$\lim_{i\to\infty} \int_I f(a_i u(\varphi(t))x)\,\mathrm{d}t = \int_{t\in I} \left(\int_{y \in G/\Gamma} f(\zeta(t)h_1 y)\,\mathrm{d}\mu_H \right) \mathrm{d}t. \tag{13}$$

Moreover $A \subset h_1 H h_1^{-1}$, $N \cap h_1 H h_1^{-1} \neq \{e\}$, *and there exists* $g \in G$ *such that* $x = \pi(g)$ *and*

$$u(\varphi(t))g \in N^{-}\zeta(t)h_1 H, \quad \forall t \in I. \tag{14}$$

Remark 1.2. Suppose we are given a convergent sequence $x_i \to x$ in G/Γ. We consider (13) for x_i in place of x in the statement of Theorem 1.8. Then the limiting distribution depends on the choice of the sequence $\{a_i\}$. We can still conclude that the analogue of (13) holds after passing to a subsequence.

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2. Non-divergence of translated measures

Let $\varphi : I \to \mathbb{R}^{n-1}$ be a nonconstant analytic map. Let $\{a_i\} \subset A^+$ be a sequence such that $\alpha(a_i) \xrightarrow{i\to\infty} \infty$. Let $x_i \xrightarrow{i\to\infty} x$ be a convergent sequence in G/Γ. For each $i \in \mathbb{N}$, let μ_i be the measure on G/Γ defined by

$$\int_{G/\Gamma} f \,\mathrm{d}\mu_i := \frac{1}{|I|}\int_I f(a_i u(\varphi(t))x_i)\,\mathrm{d}t \quad \forall f \in \mathrm{C_c}(G/\Gamma). \tag{15}$$

This section is devoted to the proof of the following:

Theorem 2.1. *Given* $\epsilon > 0$ *there exists compact set* $\mathcal{K} \subset G/\Gamma$ *such that* $\mu_i(\mathcal{K}) \geq 1 - \epsilon$ *for all* $i \in \mathbb{N}$.

We begin with some notation. Let $d = \dim N$, $\mathfrak{g}$ denote the Lie algebra of G, and $V = \wedge^d \mathfrak{g}$. Let $\boldsymbol{p} \in \wedge^d \mathfrak{n} \setminus \{0\}$. Consider the $\wedge^d$ Ad-action of G on V.

2.1. (C,α)**-good family.** Let $\mathscr{F}$ be the $\mathbb{R}$-span of the coordinate functions of the map $\Upsilon : I \to \mathrm{End}(V)$ given by $\Upsilon(t) = \wedge^d \mathrm{Ad}(u(\varphi(t)))$ for all $t \in I$.

Fix $t_0 \in I$ and let $\mathcal{E}$ be the smallest subspace of $\mathrm{End}(V)$ such that $\Upsilon(I) \subset \mathcal{E} + \Upsilon(t_0)$. Then $\Upsilon(I) \subset \mathcal{E} + \Upsilon(t)$ for all $t \in I$. For any $t \in I$, we have $\mathcal{E}_t := \mathrm{span}\{\Upsilon^{(k)}(t) : k \geq 1\} \subset \mathcal{E}$, where $\Upsilon^{(k)}(t)$ denotes the k-th derivative at t. Since Υ is an analytic function, we have $\Upsilon(I) \subset \Upsilon(s) + \mathcal{E}_s$. Therefore $\mathcal{E} \subset \mathcal{E}_s$. Hence $\mathcal{E}_s = \mathcal{E}$ for all $s \in I$.

Therefore by [6, Proposition 3.4], applied to the function $t \mapsto \Upsilon(t) - \Upsilon(t_0)$ from I to $\mathcal{E}$, there exist constants $C > 0$ and $\alpha > 0$ such that the

family $\mathscr{F}$ consists of (C,α)-*good functions*; that is, for any subinterval $J \subset I$, $\xi \in \mathcal{F}$ and $r > 0$,

$$|\{t \in J : |\xi(t)| < r\}| \leq C \left(\frac{r}{\sup_{t\in J} |\xi(t)|} \right)^{\alpha} |J|. \tag{16}$$

It may be noted that, since I is compact, by the result quoted above, a priori (16) holds only for subintervals J with $|J|$ smaller than a fixed constant depending on Υ and I. Then by a straightforward argument using a finite well-overlapping covering of I by short intervals of fixed length, and applying the above inequality successively, we can choose a much larger C such that (16) for all subintervals $J \subset I$.

Now we fix a norm $\|\cdot\|$ on V. Then given any $\epsilon > 0$ and $r > 0$, there exists $R > 0$ such that for any $h_1, h_2 \in G$ and an interval $J \subset I$, one of the following holds:

I) $\sup_{t\in J}\|h_1 u(\varphi(t))h_2\boldsymbol{p}\| < R$.

II)

$$|\{t \in J : \|h_1 u(\varphi(t))h_2\boldsymbol{p}\| \leq r\}| \leq \epsilon\,|\{t \in J : \|h_1 u(\varphi(t))h_2\boldsymbol{p}\| \leq R\}|$$

.

Proposition 2.2 ([1]). *There exists a finite set $\Sigma \subset G$ such that $\Gamma\Sigma\boldsymbol{p}$ is a discrete subset of V, and the following holds: Given $\epsilon > 0$ and $R > 0$, there exists a compact set $\mathcal{K} \subset G/\Gamma$ and such that for any $h_1, h_2 \in G$, and a subinterval $J \subset I$, one of the following holds:*

I) *There exists $\gamma \in \Gamma$ and $\sigma \in \Sigma$ such that*

$$\sup_{t\in J}\|h_1 u(\varphi(t))h_2\sigma\gamma\boldsymbol{p}\| < R.$$

II) $|t \in J : \pi(h_1 u(\varphi(t))h_2) \in \mathcal{K}\}| \geq (1-\epsilon)|J|$.

In the above proposition, $(\sigma N \sigma^{-1}) \cap \Gamma$ is a cocompact lattice in $\sigma N \sigma^{-1}$ for each $\sigma \in \Sigma$.

Now we will make an observation which will allow us to prove that the possibility (I) in the conclusion of the above proposition will not hold in the situation of our interest.

2.2. **Basic lemma.** Consider a linear representation of $\mathrm{SL}(2,R)$ on a finite dimensional vector space V. Let $a = \begin{pmatrix} \alpha & \\ & \alpha^{-1} \end{pmatrix}$ for some $\alpha > 1$, and define

$$\begin{aligned} V^{+} &= \{v \in V : a^{-k}v \to \infty \text{ as } k \to \infty\} \\ V^{0} &= \{v \in V : av = v\} \\ V^{-} &= \{v \in V : a^{k}v \to 0 \text{ as } k \to \infty\}. \end{aligned} \tag{17}$$

Then any $v \in V$ can be uniquely expressed as $v = v^+ + v^0 + v^-$, where $v^\pm \in V^\pm$ and $v^0 \in V^0$. We also write $V^{+0} = V^+ + V^0$, and $V^{0-} = V^0 + V^-$. Let $q^+ : V \to V^+$, $q^0 : V \to V^0$, $q^{+0} : V \to V^{+0}$, and $q^{0-} : V \to V^{0-}$ denote the projections $q^+(v) = v^+$, $q^0(v) = v^0$, $q^{+0}(v) = v^{+0} := v^+ + v^0$, and $q^{0-}(v) = v^{0-} := v^0 + v^-$ for all $v \in V$. We consider the Euclidean norm on V such that V^+, V^0 and V^- are orthogonal.

Lemma 2.3. *Let $\boldsymbol{u} = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ for some $t \neq 0$. Then there exists a constant $\kappa = \kappa(t) > 0$ such that*

$$\max\{\|v^+\|, \|(\boldsymbol{u}v)^{+0}\|\} \geq \kappa\|v\|, \quad \forall v \in V. \tag{18}$$

Proof. Since it is enough to prove the result for each of the $\mathrm{SL}(2,\mathbb{R})$-irreducible subspace of V. Therefore without loss of generality we may assume that $\mathrm{SL}(2,\mathbb{R})$ acts irreducibly on V.

Let $m = \dim V - 1$. Then $m = 2r-1$ or $m = 2r$ for some $r \in \mathbb{N}$. Consider the associated representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ on V. Let $\boldsymbol{e} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $\boldsymbol{h} = \left(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right)$, and $\boldsymbol{f} = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ denote the standard $\mathfrak{sl}_2$-triple. Then there exists a basis of V consisting of elements $v_0, v_1, \dots, v_m$ such that

$$\boldsymbol{h}v_k = (m-2k)v_k, \quad \text{and} \quad \boldsymbol{e}v_k = kv_{k-1}, \quad \forall 0 \leq k \leq m,$$

where $v_{-1} = 0$. Then

$$V^{+0} = \mathrm{span}\{v_0, \dots, v_{m-r}\} \quad \text{and} \quad V^{0-} = \mathrm{span}\{v_r, \dots, v_m\}. \tag{19}$$

Since $\boldsymbol{u} = \exp(t\boldsymbol{e})$, we have

$$\boldsymbol{u}v_k = \sum_{l=0}^{k} \binom{k}{l} t^{k-l} v_l, \quad 0 \leq k \leq m.$$

Let A denote the restriction of the map $\boldsymbol{u}$ from V^+ to V^+ with respect to the basis $\{v_0, \dots, v_{r-1}\}$. Let B denote the matrix of the map $q^{+0} \circ \boldsymbol{u} : V^{0-} \to V^{+0}$ with respect to the basis given by (19).

Next we want to show that B is invertible. We write $b_{k,l} = t^{k-l}\binom{k}{l}$ for $r \leq k \leq m$ and $0 \leq l \leq m-r$. And for any $r \leq m_1 \leq m$, we consider the $(m_1 - r + 1) \times (m_1 - r + 1)$-matrix

$$B(m_1, r) = \big(b_{k,l}\big)_{\substack{r \leq k \leq m_1 \\ 0 \leq l \leq m_1 - r}}.$$

Then $B = B(m, r)$. In view of the binomial relations

$$\binom{k+1}{l+1} - \binom{k}{l+1} = \binom{k}{l} \quad \text{and} \quad b_{k+1,l+1} - tb_{k,l+1} = b_{k,l},$$

we apply the row operations $R_{k+1} - tR_k$, successively, in the order $k = (m_1 - 1), \ldots, 1$. We obtain that

$$\det B(m_1, r) = t^r \det B(m_1 - 1, r).$$

Since $\det B(r,r) = t^r$, we get

$$\det B = \det B(m, r) = t^{r(m-r+1)}.$$

Since $t \neq 0$, B is invertible.

Now

$$\|(\boldsymbol{u}v)^{+0}\| = \|Av^+ + Bv^{0-}\|. \tag{20}$$

Since A is a unipotent matrix, $\|A\| \geq 1$. We put

$$\kappa = (1/3)\min\{1, \|B^{-1}\|^{-1}\|A\|^{-1}\} \leq (1/3)\min\{1, \|B^{-1}\|^{-1}\}. \tag{21}$$

Now to prove (18) it is enough to consider the case when

$$\|v^+\| \leq \kappa\|v\| \leq (1/3)\|v\|. \tag{22}$$

In particular,

$$\|v^{0-}\| \geq \|v\| - \|v^+\| \geq \|v\| - (1/3)\|v\| = (2/3)\|v\|. \tag{23}$$

Then by (20), (21), (23), and (22),

$$\begin{aligned}
\|(\boldsymbol{u}v)^{+0}\| &\geq \|Bv^{0-}\| - \|Av^+\| \\
&\geq \|B^{-1}\|^{-1}\|v^{0-}\| - \|A\|\|v^+\| \\
&\geq \|B^{-1}\|^{-1}\|v^{0-}\| - \kappa\|A\|\|v\| \\
&\geq (\|B^{-1}\|^{-1} - (3/2)\kappa\|A\|)\|v^{0-}\| \\
&\geq (1/2)\|B^{-1}\|^{-1}\|v^{0-}\| \\
&\geq (1/2)\|B^{-1}\|^{-1}(2/3)\|v\| \\
&\geq \kappa\|v\|.
\end{aligned}$$

□

Corollary 2.4. *Let V be a finite dimensional normed linear space. Consider a liner representation of $G = \mathrm{SO}(n,1)$ on V, where $n \geq 2$. Let*

$$\begin{aligned}
V^+ &= \{v \in V : a^{-k}v \xrightarrow{k\to\infty} \infty,\ \forall a \in A^+\} \\
V^- &= \{v \in V : a^{k}v \xrightarrow{k\to\infty} \infty,\ \forall a \in A^+\} \\
V^0 &= \{v \in V : Av = v\}.
\end{aligned} \tag{24}$$

Then given a compact set $F \subset N \smallsetminus \{e\}$, there exists a constant $\kappa > 0$ such that for any $\boldsymbol{u} \in F$,

$$\max\{\|v^+\|, \|(\boldsymbol{u}v)^{+0}\|\} \geq \kappa\|v\|, \quad \forall v \in V. \tag{25}$$

In particular, for any $a \in A^+$, and any $\boldsymbol{u} \in F$,

$$\max\{\|av\|, \|a\boldsymbol{u}v\|\} \geq \kappa\|v\|, \quad \forall v \in V.$$

Proof. Given any $a \in A^+$ and $\boldsymbol{u} \in F$, there exist a continuous homomorphism of $\mathrm{SL}(2,\mathbb{R})$ into G such that a is the image of $\left(\begin{smallmatrix} \alpha & \\ & \alpha^{-1} \end{smallmatrix}\right)$ for some $\alpha > 1$, and $\boldsymbol{u}$ is the image of $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ for some $t \neq 0$. We apply Lemma 2.3 to obtain a constant $\kappa_1 > 0$ such that (25) holds for $\boldsymbol{u}$.

Now there exists a compact set $F_1 \subset \mathrm{Z_G}(A)$ such that any $\boldsymbol{u}_1 \in F$ is of the form $z\boldsymbol{u}z^{-1}$ for some $z \in F_1$. Also there exists a constant $\kappa_2 > 0$ such that

$$\kappa_2\|v\| \leq \|zv\| \leq \kappa_2^{-1}\|v\| \quad \forall z \in F_1,\ \forall v \in V.$$

Using this fact, we see that (25) holds for any $\boldsymbol{u}_1 \in F$ in place of $\boldsymbol{u}$ and $\kappa := \kappa_2^2\kappa_1$. $\square$

2.3. **Proof of Theorem 2.1.** Let $t_1, t_2 \in I$ be such that $\boldsymbol{u} := u(\varphi(t_2) - \varphi(t_1))^{-1} \neq e$. By Corollary 2.4 there exists $\kappa > 0$ such that

$$\sup \|a_i v\|, \|a_i \boldsymbol{u} v\| \geq \kappa\|v\|, \quad \forall v \in V. \tag{26}$$

Let a sequence $g_i \to g \in G$ be such that $\pi(g_i) = x_i$. By Proposition 2.2 $\Gamma\Sigma\boldsymbol{p}$ is discrete in V. Therefore

$$R_1 := \inf\{\|u(\varphi(t_1)g_i\gamma\sigma\boldsymbol{p}\| : \gamma \in \Gamma,\ \sigma \in \Sigma\} > 0.$$

For any $\gamma \in \Gamma$ and $\sigma \in \Sigma$, if we put $v = u(\varphi(t_1))g_i\gamma\sigma\boldsymbol{p}$ in (26), then have

$$\sup_{t_1,t_2}\{\|a_i u(\varphi(t))g_i\gamma\sigma\boldsymbol{p}\|\} \geq \kappa\|u(\varphi(t_1))g_i\gamma\sigma\boldsymbol{p}\| \geq \kappa R_1 \tag{27}$$

Now given $\epsilon > 0$, and we obtain a compact set $\mathcal{K} \subset G/\Gamma$ such that the conclusion of Proposition 2.2 holds for $R = (1/2)\kappa R_1$. Then by (27), for any $i \in \mathbb{N}$, the possibility (I) in the conclusion of Proposition 2.2 does not hold for $h_1 = a_i$, $h_2 = g_i$. Therefore the possibility (II) of Proposition 2.2 must hold for all i. Thus Theorem 2.1 follows. $\square$

We obtain the following immediate consequence of Theorem 2.1:

Corollary 2.5. *After passing to a subsequence, $\mu_i \to \mu$ in the space of probability measures on G/Γ with respect to the weak*-topology; that is,*

$$\lim_{i\to\infty}\int_{G/\Gamma} f \,\mathrm{d}\mu_i = \int_{G/\Gamma} f\,\mathrm{d}\mu, \quad \forall f \in \mathrm{C_c}(G/\Gamma).$$

3. Invariance under a unipotent flow

Let $I = [a,b] \subset \mathbb{R}$ with $a < b$. Let $\varphi : I \to \mathbb{R}^{n-1}$ be a C^2-curve such that $\dot{\varphi}(t) \neq 0$ for all $t \in I$, where $\dot{\varphi}(t)$ denotes the tangent to the curve φ at t. Fix $w_0 \in \mathbb{R}^{n-1} \smallsetminus \{0\}$, and define

$$W = \{u(tw_0) : t \in \mathbb{R}\}.$$

Consider the $\mathrm{Z}_{\mathrm{G}}(A)$-action on $\mathbb{R}^{n-1}$ via the correspondence $u(zv) = zu(v)z^{-1}$ for all $v \in \mathbb{R}^{n-1}$. Then $\mathrm{Z}_{\mathrm{G}}(A) = MA$ acts transitively on $\mathbb{R}^{n-1} \setminus \{0\}$. Therefore there exists a continuous function $z : I \to \mathrm{Z}_{\mathrm{G}}(A)$ such that

$$z(t)\dot{\varphi}(t) = w_0, \quad \forall t \in I. \tag{28}$$

Let a sequence $\{a_i\}_{i\in\mathbb{N}} \subset A^+$ be such that $\alpha(a_i) \to \infty$ as $i \to \infty$. Let $x_i \to x$ a convergent sequence in G/Γ. For each $i \in \mathbb{N}$, let λ_i be the probability measure on G/Γ such that

$$\int_{G/\Gamma} f d\lambda_i = \frac{1}{|I|}\int_{t\in I} f(z(t)a_i u(\varphi(t))x_i)\,\mathrm{d}t, \quad \forall f \in \mathrm{C_c}(G/\Gamma). \tag{29}$$

Since $z(I)$ is compact, by Theorem 2.1, there exists a probability measure λ on G/Γ such that, after passing to a subsequence, $\lambda_i \to \lambda$ as $i \to \infty$, in the space of finite measures on G/Γ with respect to the weak*-topology.

Theorem 3.1. *The measure λ is W-invariant.*

Proof. We will use the notation $\eta_1 \overset{\epsilon}{\approx} \eta_2$ to say $|\eta_1 - \eta_2| \leq \epsilon$.

Let $f \in \mathrm{C_c}(G/\Gamma)$ and $\epsilon > 0$ be given. Let Ω be a neighbourhood of e in G such that

$$f(\omega y) \overset{\epsilon}{\approx} f(y), \quad \forall \omega \in \Omega^2 \text{ and } \forall y \in G/\Gamma. \tag{30}$$

Let $t_0 \in \mathbb{R}$. Let $t \in I = [a, b]$ and $i \in \mathbb{N}$. By (28)

$$u(t_0 w_0)z(t)a_i = z(t)a_i u(\alpha(a_i)^{-1}t_0 z(t)^{-1} \cdot w_0) = z(t)a_i u(\xi_i\dot{\varphi}(t)), \tag{31}$$

where $\xi_i := \alpha(a_i)^{-1}t_0$. Since φ is a C^2-map,

$$\varphi(t + \xi_i) = \varphi(t) + \xi_i\dot{\varphi}(t) + \epsilon_i(t), \tag{32}$$

where by Taylor's formula, there exists a constant $M > 0$ such that

$$|\epsilon_i(t)| \leq M|\xi_i|^2 \leq (M|t_0|^2)\alpha(a_i)^{-2}, \quad \forall t \in [a, b]. \tag{33}$$

As $i \to \infty$, we have $\alpha(a_i) \to \infty$, and hence $\xi_i \to 0$ and $\alpha(a_i)\epsilon_i(t) \to 0$. Since $t \mapsto z(t)$ is continuous, there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$,

$$z(t + \xi_i)z(t)^{-1} \in \Omega \quad \text{and} \quad u(z(t) \cdot (\alpha(a_i)\epsilon_i(t))) \in \Omega. \tag{34}$$

Therefore

$$\begin{aligned}
& z(t+\xi_i)a_i u(\varphi(t+\xi_i)) \\
= \;& (z(t+\xi_i)z(t)^{-1})z(t)a_i u(\varphi(t) + \xi_i\dot{\varphi}(t) + \epsilon_i(t)), \quad \text{by (32)} \\
\in \;& \Omega u(z(t)\cdot(\alpha(a_i)\epsilon_i(t)))z(t)a_i(u(\varphi(t) + \xi_i\dot{\varphi}(t))) \\
\subset \;& \Omega^2 z(t)a_i u(\xi_i\dot{\varphi}(t))u(\varphi(t)), \quad \text{by (34)} \\
\subset \;& \Omega^2 u(t_0 w_0)z(t)a_i u(\varphi(t)), \quad \text{by (31).}
\end{aligned} \tag{35}$$

Therefore by (30)

$$f(z(t+\xi_i)a_i u(\varphi(t+\xi_i)x_i) \stackrel{\epsilon}{\approx} f(u(t_0 w_0)z(t)a_i u(\varphi(t))x_i). \tag{36}$$

Hence

$$\begin{aligned} & \int_a^b f(z(t)a_i u(\varphi(t))x_i)\,\mathrm{d}t \\ \stackrel{\xi_i \sup|f|}{\approx} & \int_a^{b-\xi_i} f(z(t+\xi_i)a_i u(\varphi(t+\xi_i))x_i)\,\mathrm{d}t \\ \stackrel{\epsilon|I|}{\approx} & \int_a^{b-\xi_i} f(u(t_0 w_0)z(t)a_i u(\varphi(t+\xi_i))x_i)\,\mathrm{d}t, \quad \text{by (36)} \\ \stackrel{\xi_i \sup|f|}{\approx} & \int_a^b f(u(s_0 w_0)z(t)a_i u(\varphi(t))x_i)\,\mathrm{d}t. \end{aligned} \tag{37}$$

Therefore, since $\epsilon > 0$ is chosen arbitrarily, and $\xi_i \to 0$ as $i \to 0$,

$$\int_{G/\Gamma} f(u(s_0 w_0)y)\,\mathrm{d}\lambda(y) = \int_{G/\Gamma} f(y)\,\mathrm{d}\lambda(y). \tag{38}$$

□

The above simplification of the original proof given in arXiv:0708.4093v1 is based on referee's suggestions.

4. Dynamical behaviour of translated trajectories near singular sets

Let the notation be as in the previous section. We will further assume that $\varphi : I \to \mathbb{R}^{n-1}$ is an analytic function. In this case we will further observe that the function $z : I \to \mathrm{Z_G}(A)$ such that $z(t)\dot{\varphi}(t) = w_0$ for all $t \in I$ is also an analytic function. Given a convergent sequence $x_i \to x$ in G/Γ, we obtain a sequence of measures $\{\lambda_i : i \in \mathbb{N}\}$ on G/Γ as defined by (29). Due to Theorem 2.1, by passing to a subsequence we will assume that $\lambda_i \to \lambda$ as $i \to \infty$, where λ is a probability measure on G/Γ. By Theorem 3.1, λ is invariant under the action of the one-parameter subgroup $W = \{u(sw_0) : s \in \mathbb{R}\}$. We would like to describe the measure λ using the description of ergodic invariant measures for unipotent flows on homogeneous spaces due to Ratner [11]. We begin with some notation.

Let $\mathscr{H}$ denote the collection of analytic subgroups H of G such that $H \cap \Gamma$ is a lattice in H, and a unipotent one-parameter subgroup of H acts ergodically with respect to the H-invariant probability measure on $H/H \cap \Gamma$. Then $\mathscr{H}$ is a countable collection [14, 11].

For $H \in \mathscr{H}$, define

$$\begin{aligned} N(H,W) &= \{g \in G : g^{-1}Wg \subset H\} \qquad \text{and} \\ S(H,W) &= \textstyle\bigcup_{\substack{F \in \mathscr{H} \\ F \subsetneq H}} N(F,W). \end{aligned}$$

Then by [9, Lemma 2.4]

$$\pi(N(H,W)\setminus S(H,W)) = \pi(N(H,W))\setminus \pi(S(H,W)). \tag{39}$$

Then by Ratner's theorem [11], as explained in [9, Theorem 2.2]:

Theorem 4.1 (Ratner). *Given the W-invariant probability measure λ on G/Γ, there exists $H\in\mathscr{H}$ such that*

$$\lambda(\pi(N(H,W))>0 \quad \textit{and} \quad \lambda(\pi(S(H,W))=0. \tag{40}$$

Moreover almost every W-ergodic component of λ on $\pi(N(H,W))$ is a measure of the form $g\mu_H$, where $g\in N(H,W)\setminus S(H,W)$, μ_H is a finite H-invariant measure on $\pi(H)\cong H/H\cap\Gamma$, and $g\mu_H(E):=\mu(g^{-1}E)$ for all Borel sets $E\subset G/\Gamma$.

For $d=\dim H$, let $V=\wedge^d\mathfrak{g}$, and consider the $\wedge^d$ Ad-action of G on V. Fix $p_H=\wedge^d\mathfrak{h}\setminus\{0\}$.

We recall some facts from [9, §3]: For any $g\in \mathrm{N_G}(H)$, $gp_H=\det(\mathrm{Ad}\,g|_{\mathfrak{h}})p_H$. Hence the stabilizer of p_H in G equals

$$\mathrm{N^1_G}(H):=\{g\in\mathrm{N_G}(H):\det((\mathrm{Ad}\,g)|_{\mathfrak{h}})=1\}.$$

Since, $(\Gamma\cap\mathrm{N_G}(H))\pi(H)=\pi(H)$, we have $(\Gamma\cap\mathrm{N_G}(H))p_H=p_H$ or $(\Gamma\cap\mathrm{N_G}(H))p_H=\{p_H,-p_H\}$. In the former case we put $\bar V=V$ and in the later case we put $\bar V=V/\{\pm1\}$. For any $v\in V$, we denote by $\bar v$ the image of v in $\bar V$, and define the action of $g\in G$ by $g\cdot\bar v:=\bar{gv}$. We define $\eta(g)=g\bar p_H$ for all $g\in G$.

Proposition 4.2 ([2]). *$\eta(\Gamma)=\Gamma\cdot\bar p_H$ is a discrete subset of $\bar V$.* □

Let $\mathcal{A}=\{\bar v\in\bar V: v\wedge w_0=0\in\wedge^{d+1}\mathfrak{g}\}$. Then $\mathcal{A}$ is the image of a linear subspace of V. We observe that

$$N(H,W)=\eta^{-1}(\mathcal{A}). \tag{41}$$

Given any compact set $\mathcal{D}\subset\mathcal{A}$, we define

$$S(\mathcal{D})=\{g\in\eta^{-1}(\mathcal{D}):\eta(g\gamma)\in\mathcal{D}\quad\text{for some }\gamma\in\Gamma\setminus\mathrm{N_G}(H)\}.$$

Proposition 4.3 ([9, Proposition 3.2]). *(1) $S(\mathcal{D})\subset S(H,W)$ and $\pi(S(\mathcal{D}))$ is closed in G/Γ. (2) For any compact set $\mathcal{K}\subset G/\Gamma\setminus\pi(S(\mathcal{D}))$, there exists a neighbourhood Φ of $\mathcal{D}$ in $\bar V$ such that for any $g\in G$ and $\gamma_1,\gamma_2\in\Gamma$:*

$$\textit{if } \pi(g)\in\mathcal{K} \textit{ and } \{\eta(g\gamma_1),\eta(g\gamma_2)\}\subset\overline{\Phi}, \textit{ then } \eta(\gamma_1)=\eta(\gamma_2), \tag{42}$$

where $\overline{\Phi}$ denotes the closure of Φ in $\bar V$.

4.1. (C,α)**-good family.** Let $\mathscr{F}$ denote the $\mathbb{R}$-span of all the coordinate functions of the maps $t \mapsto (\wedge^d \operatorname{Ad})(z(t)u(\varphi(t)))$ from I to $\mathrm{GL}(V)$. As explained in §2.1, by [6, Proposition 3.4], the family $\mathscr{F}$ is '(C,α)-good' for some $C > 0$ and $\alpha > 0$; that is, for any subinterval $J \subset I$, $\xi \in \mathscr{F}$, and $r > 0$,

$$|\{t \in J : |\xi(t)| < r\}| < C \left(\frac{r}{\sup_{t\in J} |\xi(t)|}\right)^{\alpha} |J|. \tag{43}$$

Proposition 4.4 (Cf. [2])**.** *Given a compact set $\mathcal{C} \subset \mathcal{A}$ and $\epsilon > 0$, there exists a compact set $\mathcal{D} \subset \mathcal{A}$ containing $\mathcal{C}$ such that given any neighbourhood Φ of $\mathcal{D}$ in $\bar{V}$, there exists a neighbourhood Ψ of $\mathcal{C}$ in $\bar{V}$ contained in Φ such that for any $h \in G$, any $v \in \bar{V}$ and any interval $J \subset I$, one of the following holds:*

I) *$hz(t)u(\varphi(t))v \in \Phi$ for all $t \in J$.*
II) *$|\{t \in J : hz(t)u(\varphi(t))v \in \Psi\}| \le \epsilon|\{t \in J : hz(t)u(\varphi(t))v \in \Phi\}|$.*

Proof. The argument in the proof of [2, Proposition 4.2] goes through with straightforward changes. Since $\mathcal{A}$ is the image of a linear subspace of V, one can describe the neighbourhoods of subsets of $\mathcal{A}$ in $\bar{V}$ via linear functionals. Further, one uses the property (43) of the functions in $\mathscr{F}$ instead of [2, Lemma 4.1] in the proof. □

4.2. **Linear presentation of dynamics in thin neighbourhoods of singular sets.** Now let C be any compact subset of $N(H,W) \smallsetminus S(H,W)$. Let an $\epsilon > 0$ be given. We apply Proposition 4.4 to $\mathcal{C} := \eta(C) \subset \mathcal{A}$, and obtain a compact set $\mathcal{D} \subset \mathcal{A}$. By (39), since $\pi(C)$ is a compact subset $G/\Gamma \smallsetminus \pi(\mathcal{S}(\mathcal{D}))$. We choose a compact set $\mathcal{K} \subset G/\Gamma \smallsetminus \pi(\mathcal{S}(\mathcal{D}))$ such that $\pi(C)$ is contained in the interior of $\mathcal{K}$. Then we take any neighbourhood Φ_1 of D in $\bar{V}$. By Proposition 4.3, there exists an open neighbourhood Φ of $\mathcal{D}$ contained in Φ_1 such that property (42) holds. Now we obtain a neighbourhood Ψ of $\mathcal{C}$ in $\bar{V}$ such that the conclusion of Proposition 4.4 holds. Let

$$\mathcal{O} := \pi(\eta^{-1}(\Psi)) \cap \mathcal{K}. \tag{44}$$

Then $\mathcal{O}$ is a neighbourhood of $\pi(C)$ in G/Γ.

Proposition 4.5 (Cf. [9])**.** *For any $h_1, h_2 \in G$, and for any subinterval $J \subset I$, one of the following holds:*

a) *There exists $\gamma \in \Gamma$ such that $\eta(h_1 z(t)u(\varphi(t))h_2\gamma) \in \Phi$, $\forall t \in J$.*
b) *$|\{t \in J : \pi(h_1 z(t)u(\varphi(t))h_2) \in \mathcal{O}\}| \le (2\epsilon)|J|$.*

Proof. Suppose that the possibility (a) does not hold for the given J.

Let $\psi(t) := h_1 z(t)u(\varphi(t))h_2$ for all $t \in I$. Let

$$J^* = \{t \in J : \pi(\psi(t)) \in \mathcal{O}\}. \tag{45}$$

Take any $t \in J^*$. By the choice of Φ with property (42), there exists a unique $v_t \in \eta(\Gamma)$ such that $\psi(t)v_t \in \overline{\Phi}$; and hence due to (44) and (45) we have $\psi(t)v_t \in \Psi$. Let $J(t)$ be the largest subinterval of J containing t such that

$$\psi(s)v_t \in \overline{\Phi}, \quad \forall s \in J(t). \tag{46}$$

By (42) and (46), we have

$$v_s = v_t, \text{ and hence } \psi(s)v_t = \psi(s)v_s \in \Psi, \quad \forall s \in J^* \cap J(t). \tag{47}$$

Since the possibility (a) does not hold for J, by our choice of $J(t)$, we have that $J(t)$ contains one of its end-points, say s_e, and $\psi(s_e)v_t \notin \Phi$. Thus $\psi(J(t))v_t \not\subset \Phi$. Therefore by Proposition 4.4, in view of (44) and (47), we deduce that

$$|J^* \cap J(t)| \leq \epsilon |J(t)|. \tag{48}$$

Due to (47), $J(s) = J(t)$ for all $s \in J^* \cap J(t)$. Therefore there exists a countable set $\mathcal{J}^* \subset J^*$ such that

$$J^* \subset \textstyle\bigcup_{t \in \mathcal{J}^*} J(t), \tag{49}$$

and if $t_1 \neq t_2$ in $\mathcal{J}^*$ then $t_1 \notin J(t_2)$.

In particular, if $t_1 < t_2$ in $\mathcal{J}^*$ then $J(t_1) \cap J(t_2) \subset (t_1, t_2)$. Therefore if $t_1 < t_2 < t_3$ in $\mathcal{J}^*$, then

$$J(t_1) \cap J(t_2) \cap J(t_3) = \emptyset.$$

Hence

$$\sum_{t \in \mathcal{J}^*} |J(t)| \leq 2 |\textstyle\bigcup_{t \in \mathcal{J}^*} J(t)|. \tag{50}$$

Now by (48), (49) and (50),

$$|\mathcal{J}^*| \leq \epsilon \sum_{t \in \mathcal{J}^*} |J(t)| \leq (2\epsilon)|J|.$$

□

4.3. **Algebraic consequences of positive limit measure on singular sets.** Let $\{a_i\} \subset A$ and $x_i \xrightarrow{i \to \infty} x$ be the sequences which are involved in the definition of λ_i, see (29).

Let $V^+ = \{v \in V : a_i^{-1} v \xrightarrow{i \to \infty} 0\}$, $V^0 = \{v \in V : Av = v\}$, and $V^- = \{v \in V : a_i v \xrightarrow{i \to \infty} 0\}$. Then $V = V^+ \oplus V^0 \oplus V^-$. Let $q^+ : V \to V^+$ and $q^{+0} : V \to V^+ + V^0$ denote the associated projections. Let $\bar{V}^+$, $\bar{V}^0$ and $\bar{V}^-$ denote the projections of V^+, V^0 and V^-, on $\bar{V}$ respectively. The sets $V^{\pm}$ do not change if we pass to a subsequence of $\{a_i\}$.

We recall that after passing to a subsequence, $\lambda_i \to \lambda$ in the space of probability measures on G/Γ, and by Theorem 3.1 and Theorem 4.1 there exists $H \in \mathscr{H}$ such that

$$\lambda(\pi(N(H,W) \smallsetminus S(H,W))) > 0. \tag{51}$$

The goal of this section is to analyze this condition using Proposition 4.5 and Corollary 2.4 to obtain its following algebraic consequence.

Proposition 4.6. *Let $g \in G$ be such that $\pi(g) = x$. Then there exists $\gamma \in \Gamma$ such that*

$$\eta(z(t)u(\varphi(t))g\gamma) \subset \bar{V}^0 + \bar{V}^-, \quad \forall t \in I. \tag{52}$$

Proof. By (51) there exists a compact set $C \subset N(H,W) \smallsetminus S(H,W)$ such that $\lambda(\pi(C)) > c_0 > 0$ for some constant $c_0 > 0$. We fix $0 < \epsilon < c_0/2$, and obtain the compact sets $\mathcal{D} \subset \mathcal{A}$ and $\mathcal{K} \subset G/\Gamma$ as in §4.2. Next we choose any neighbourhood Φ_1 of $\mathcal{D}$ in $\bar{V}$, and obtain a neighbourhood Ψ of $\eta(C)$ as in §4.2. Let $i_1 \in \mathbb{N}$ be such that if we put $\mathcal{O} := \pi(\eta^{-1}(\Psi)) \cap \mathcal{K}$, then

$$\lambda_i(\mathcal{O}) > c_0 \quad \text{for all } i \geq i_1. \tag{53}$$

Since $x_i \xrightarrow{i\to\infty} x$ and $\pi(g) = x$, there exists a convergent sequence $g_i \xrightarrow{i\to\infty} g$ in G such that $\pi(g_i) = x_i$ for all $i \in \mathbb{N}$. By (53) and (29), since $z(t) \in \mathrm{Z}_{\mathrm{G}}(A)$,

$$|\{t \in I : \pi(a_i z(t)u(\varphi(t))g_i) \in \mathcal{O}\}| > c_0|I|, \quad \forall i \geq i_1. \tag{54}$$

We apply Proposition 4.5 for $h_1 = a_i$, $h_2 = g_i$, and $J = I$. Then since $c_0 > 2\epsilon$, by (54), the possibility (b) in the conclusion of Proposition 4.5 does not hold, and hence possibility (a) of the proposition must hold; that is, there exists $\gamma_i \in \Gamma$ such that

$$\eta(a_i z(t)u(\varphi(t))g_i\gamma_i) \in \Phi \subset \Phi_1, \quad \forall i \geq i_1. \tag{55}$$

We choose a decreasing sequence of neighbourhoods Φ_k of $\mathcal{D}$ in $\bar{V}$ be such that $\bigcap_{k\in\mathbb{N}} \Phi_k = \mathcal{D}$, and apply the above argument for each Φ_k in place of Φ_1. We then obtain sequences $i_k \xrightarrow{k\to\infty} \infty$ in $\mathbb{N}$ and $\{\gamma_k\}$ in Γ such that

$$a_{i_k} z(t)u(\varphi(t))\eta(g_{i_k}\gamma_k) \in \Phi_k, \quad \forall t \in I, \ \forall k \in \mathbb{N}.$$

Since $\{z(t) : t \in I\}$ is contained in a compact set, there exists $R > 0$ such that $z(I)^{-1}\Phi_1$ is contained in $B(R)$, the ball of radius R centered at 0 in $\bar{V}$. Thus

$$\|a_{i_k} u(\varphi(t))\eta(g_{i_k}\gamma_k)\| \leq R, \qquad \forall t \in I. \tag{56}$$

Fix any $t_1 \in I$. Since φ is a nonconstant function, by Corollary 2.4, there exists a constant $\kappa > 0$ such that

$$\sup_{t\in I} \|q^{+0}(u(\varphi(t))v)\| \geq \kappa \|u(\varphi(t_1))v\|, \quad \forall v \in V. \tag{57}$$

For all $v \in V$, define $\|\bar{v}\| := \|v\|$, and let $q^{+0}(\bar{v})$ and $q^{+}(\bar{v})$ denote the images of $q^{+0}(v)$ and $q^{+}(v)$ in $\bar{V}$, respectively. Let $t \in I$. Then

$$\begin{aligned}
&\|q^{+0}(u(\varphi(t)\eta(g_{i_k}\gamma_k)))\| \\
&\leq \|a_{i_k} q^{+0}(u(\varphi(t)\eta(g_{i_k}\gamma_k)))\| \\
&\leq \|a_{i_k} u(\varphi(t))\eta(g_{i_k}\gamma_k)\| \\
&\leq R, \qquad \text{(by (56))}.
\end{aligned}$$

Therefore by (57),

$$\|\eta(g_{i_k}\gamma_k)\| \leq \kappa^{-1}\|u(\varphi(t_1))^{-1}\|. \tag{58}$$

Since $\eta(\Gamma)$ is discrete and $g_{i_k} \xrightarrow{k\to\infty} g$, due to (58), the set $\{\eta(\gamma_k) : k \in \mathbb{N}\}$ is finite. Therefore by passing to a subsequence, there exists $\gamma \in \Gamma$ such that $\eta(\gamma_k) = \eta(\gamma)$ for all $k \in \mathbb{N}$ and hence

$$\eta(a_{i_k} z(t) u(\varphi(t)) g_{i_k}\gamma) \in \Phi_k, \ \forall k \in \mathbb{N}. \tag{59}$$

For each $k \in \mathbb{N}$, if $w_k^+ = q^+(\eta(z(t)u(\varphi(t))g_{i_k}\gamma)) \in \bar{V}^+$, then by (59) we have $\limsup_{k\to\infty}\|a_{i_k} w_k^+\| < \infty$. Since $\alpha(a_{i_k}) \xrightarrow{k\to\infty} \infty$, we conclude that $w_k^+ \xrightarrow{k\to\infty} 0$. Since $g_{i_k} \xrightarrow{k\to\infty} g$, we have

$$q^+(\eta(z(t)u(\varphi(t)g\gamma))) = \lim_{k\to\infty} q^+(\eta(z(t)u(\varphi(t)g_{i_k}\gamma))) = \lim_{k\to\infty} w_k^+ = 0.$$

Therefore (52) follows. $\square$

In order to derive group theoretic consequences of condition (52) we will need the following observation.

Lemma 4.7. *If $\eta(h_1), \eta(h_2) \in \bar{V}^0 \cap \mathcal{A}$ for some $h_1, h_2 \in G$, then*

$$A \subset h_1 H h_1^{-1}, \quad \mathrm{N_G}(H)^0 \subset \mathrm{Z_G}(H)H, \quad \textit{and} \quad h_2 \in (M \cap \mathrm{Z_G}(W))h_1 H.$$

Also $h_1 H h_1^{-1}$ contains a cocompact normal subgroup containing A which is conjugate to $\mathrm{SO}(m,1)$ in G, where $2 \leq m \leq n$.

Proof. Since A is connected, $A \subset h_j \mathrm{N^1_G}(H) h_j^{-1}$, $j = 1,2$. Let H_1 denote the Zariski closure of H in G. Then $A \subset h_j \mathrm{N^1_G}(H_1)h_j^{-1}$, $j = 1,2$. If U_1 denotes the unipotent radical of H_1, then

$$A \subset h_j \mathrm{N^1_G}(U_1) h_j^{-1}, \quad j = 1,2. \tag{60}$$

Any nontrivial unipotent element of G is contained in a unique maximal unipotent subgroup of G. Therefore either $h_1 U_1 h_1^{-1} \subset N$ or $h_1 U_1 h_1^{-1} \subset$

N^- (recall (9)). Therefore in view of (60), we conclude that $U_1 = \{e\}$. Therefore $\mathrm{N_G}(H_1)$ is reductive, and hence $\mathrm{N_G}(H) = \mathrm{N_G}(H_1)$ is reductive. Therefore $\mathrm{N_G}(H)^0 = \mathrm{Z_G}(H)^0 H \subset \mathrm{N^1_G}(H)$. Since $\mathrm{N_G}(H)$ contains nontrivial unipotent elements, its maximal semisimple factor contains an $\mathbb{R}$-diagonalizable subgroup. Therefore, since G is of $\mathbb{R}$-rank one, $\mathrm{Z_G}(H)$ is compact. In particular, $A \subset h_j H h_j^{-1}$, $j = 1, 2$.

Being a reductive subgroup of $\mathrm{SO}(n,1)$ containing a nontrivial unipotent element, each $h_j H h_j^{-1}$ is of the form $vM_1 \,\mathrm{SO}(m,1)v^{-1}$ for some $v \in N$ (see (9)), $2 \leq m \leq n$ and $M_1 \subset \mathrm{Z_G}(\mathrm{SO}(m,1)) \subset K$. Note that if $g_i^{-1} AW g_i \subset \mathrm{SO}(m,1)$ for some $g_1, g_2 \in G$, then there exists $g' \in \mathrm{SO}(m,1)$, such that $(g')^{-1}(g_1^{-1}awg_1)g' = g_2^{-1}awg_2$ for all $a \in A$ and $w \in W$. Therefore $g_2 \in \mathrm{Z_G}(AW)g_1 g' \subset (M \cap \mathrm{Z_G}(W))g_1 \,\mathrm{SO}(m,1)$. Thus in our situation, we conclude that have $h_2 \in (M \cap \mathrm{Z_G}(W))h_1 H$. □

4.4. **Algebraic description of λ.**

Proposition 4.8. *There exist an analytic map $\tilde{\xi} : I \to \mathrm{Z_G}(W) \cap M$ and an element $h_1 \in G$ such that $AWh_1 \subset h_1 H$ and the following holds: For any $f \in \mathrm{C_c}(G/\Gamma)$, we have*

$$\int_{G/\Gamma} f \,\mathrm{d}\lambda = \frac{1}{|I|} \int_{t \in I} \left(\int_{y \in \pi(H)} f(\tilde{\xi}(t) h_1 y) \,\mathrm{d}\mu_H(y) \right) \mathrm{d}t. \tag{61}$$

In particular, λ is AW-invariant. Moreover

$$u(\varphi(I)) \subset N \cap (P^- h_1 H (g\gamma)^{-1}) = N \cap (N^- M h_1 H (g\gamma)^{-1}). \tag{62}$$

Proof. Let the notation be as in the previous section. We start with a construction. Since $H \cap \Gamma$ is a lattice in H and $\mathrm{N_G}(H)/H$ is compact, we have that $(\mathrm{N_G}(H) \cap \Gamma)/(H \cap \Gamma)$ is finite. Since $\eta(\Gamma)$ is discrete, the map $\rho : G/(H \cap \Gamma) \to (G/\Gamma) \times \bar{V}$, defined by

$$\rho(h(H \cap \Gamma)) = (\pi(h), \eta(h)), \quad \forall h \in G,$$

is a continuous proper map.

By Proposition 4.6 there exists $\gamma \in \Gamma$ such that

$$\eta(z(t)u(\varphi(t))g\gamma) \subset \bar{V}^0 + \bar{V}^-, \quad \forall t \in I.$$

We put $\xi(t) := q^0(z(t)\eta(u(\varphi(t))g\gamma)) \in \bar{V}^0$ for all $t \in I$. Then $\xi : I \to \bar{V}^0$ is an analytic function, and

$$\xi(t) = \lim_{\substack{\alpha(a) \to \infty \\ a \in A^+}} \eta(az(t)u(\varphi(t))g\gamma) \in \mathcal{D}. \tag{63}$$

Thus

$$\xi(I) \subset \mathcal{A} \cap \bar{V}^0. \tag{64}$$

For each $i \in \mathbb{N}$, we define a probability measure $\tilde{\lambda}_i$ on $G/(H \cap \Gamma)$ such that for any $\tilde{f} \in \mathrm{C_c}(G/(H \cap \Gamma))$,

$$\int_{G/(H\cap\Gamma)} \tilde{f}\,\mathrm{d}\tilde{\lambda}_i = \frac{1}{|I|}\int_I \tilde{f}(a_i z(t)u(\varphi(t))g_i\gamma(H\cap\Gamma))\,\mathrm{d}t.$$

Then $\tilde{\lambda}_i$ projects onto λ_i under the quotient map $\rho_1 : G/(H\cap\Gamma) \to G/\Gamma$. Let $\rho_2 : G/(H \cap \Gamma) \to \bar{V}$ be the map defined by $\rho_2(h(H \cap \Gamma)) = \eta(h)$ for all $h \in G$. Then due to (63), the projected measures $(\rho_2)_*(\tilde{\lambda}_i)$ on $\bar{V}$ converge to a probability measure, say ν, supported on $\xi(I)$ as $i \to \infty$. Now since ρ is a proper map, we conclude that, after passing to a subsequence, as $i \to \infty$, $\tilde{\lambda}_i$ converges to a probability measure $\tilde{\lambda}$ on $G/(H \cap \Gamma)$ such that

$$(\rho_1)_*(\tilde{\lambda}) = \lambda \quad \text{and} \quad (\rho_2)_*(\tilde{\lambda}) = \nu. \tag{65}$$

Therefore by (64) and Lemma 4.7, there exists $h_1 \in N(H, W)$ such that

$$\xi(t) \in (\mathrm{Z_G}(W) \cap M)\eta(h_1), \quad \forall t \in I. \tag{66}$$

Hence

$$\mathrm{supp}(\tilde{\lambda}) \subset (\mathrm{Z_G}(W) \cap M)h_1H/(H \cap \Gamma). \tag{67}$$

Since G is an algebraic group acting linearly on V, the orbit $\eta(G)$ is open in its closure, and hence locally compact in the relative topology. Now $\mathrm{N_G}(H)^0 \subset \mathrm{Z_G}(H)H$ and $\mathrm{Z_G}(H) \subset h_1^{-1}Mh_1$ is compact. In particular, the quotient map $G/H \to \eta(G)$ given by $hH \mapsto \eta(h)$, for all $h \in G$, is a proper map with respect to the relative topology on $\eta(G) \subset \bar{V}$. Therefore due to (66), there exists an analytic map $\tilde{\xi} : I \to \mathrm{Z_G}(W) \cap M$ such that

$$\lim_{\substack{\alpha(a)\to\infty \\ a\in A^+}} az(t)u(\varphi(t))g\gamma H = \tilde{\xi}(t)h_1H, \qquad \forall t \in I. \tag{68}$$

By (65) and (67) λ is concentrated on $\pi(N(H, W))$. Then almost every normalized W-ergodic component of λ is of the form $hh_1\mu_H$ for some $h \in \mathrm{Z_G}(W) \cap M$, where μ_H is the H-invariant probability measures on $\pi(H)$. Therefore, since $hh_1\mu_H$ is A-invariant for each $h \in M$, we conclude that λ is A-invariant.

Let $\tilde{\eta} : G/(H \cap \Gamma) \to G/H$ be the quotient map. Let $\bar{\lambda} = \tilde{\eta}_*(\tilde{\lambda})$ be the projection of $\tilde{\lambda}$ on G/H. Then for any $\tilde{f} \in \mathrm{C_c}(G/H \cap \Gamma)$,

$$\int_{G/(H\cap\Gamma)} \tilde{f}\,\mathrm{d}\tilde{\lambda} = \int_{G/H}\left(\int_{y\in\pi(H)} \tilde{f}(hh_1y)\,\mathrm{d}\,\mu_H(y)\right)\mathrm{d}\bar{\lambda}(hh_1H). \tag{69}$$

By (68), $\bar{\lambda}$ is the projection of the normalized Lebesgue measure of I onto $\tilde{\xi}(I)h_1H/H$. Thus we obtain a complete description of the measure λ as in (61).

Since $h_1Hh_1^{-1}$ is a reductive subgroup of G containing A, if there exist $h' \in G$ and $h \in M$ such that

$$\lim_{\substack{\alpha(a)\to\infty\\ a\in A^+}} ah'(h_1H) = h(h_1H),$$

then $h' \in N^-h$. Hence by (68), there exists a continuous map $n^-(t) : I \to N^-$ such that

$$z(t)u(\varphi(t))g\gamma \in n^-(t)\tilde{\xi}(t)(h_1H), \quad \forall t \in I.$$

Therefore, since $Ah_1H = h_1H$, we obtain (62). $\square$

5. Proofs of results stated in the Introduction

In order to describe the limiting distributions for the sequence of measures μ_i as defined in (15) using Proposition 4.8, we make the following observation.

Lemma 5.1. *Let $\{\theta_i : I = [a,b] \to G/\Gamma\}_{i\in\mathbb{N}}$ be sequence of continuous curves, and $\{a_i\}_{\mathbb{N}} \subset A^+$ be a sequence such that $\alpha(a_i) \xrightarrow{i\to\infty} \infty$. Let $E \subset I$ be a finite set, and suppose that for each $t \in I \smallsetminus E$ there exists a probability measure λ_t on G/Γ such that the map $t \mapsto \lambda_t$ is continuous on $I \smallsetminus E$ with respect to the weak*-topology on the space of probability measures on G/Γ, and for every closed interval $J \subset I \smallsetminus E$ with nonempty interior, we have*

$$\lim_{i\to\infty} \frac{1}{|J|}\int_J f(a_i\theta_i(t))\,\mathrm{d}t = \int_{t\in J}\left(\int_{G/\Gamma} f\,\mathrm{d}\lambda_t\right)\mathrm{d}t.$$

Let $z_i \xrightarrow{i\to\infty} z$ from I to P^-, and and $w_i \xrightarrow{i\to\infty} w$ from I to G be uniformly convergent sequences of continuous functions. Then

$$\lim_{i\to\infty}\int_I f(w_i(t)a_iz_i(t)\theta_i(t))\,\mathrm{d}t = \int_I\left(\int_{G/\Gamma} f(w(t)\zeta(t)y)\,\mathrm{d}\lambda_t(y)\right)\mathrm{d}t,$$

where $\zeta(t) \in \mathrm{Z_G}(A)$ is such that $z(t) \in N^-\zeta(t)$ for all $t \in I$.

In particular if $\lambda_t = \mu_G$ for all $t \in I$, then

$$\lim_{i\to\infty}\frac{1}{|I|}\int_I f(w_i(t)a_iz_i(t)\theta_i(t))\,\mathrm{d}t = \int_{G/\Gamma} f\,\mathrm{d}\mu_G. \tag{70}$$

$\square$

Proof of Theorem 1.8. Since φ is analytic and nonconstant, the set $E := \{t \in I : \dot{\varphi}(t) = 0\}$ is finite. Let $J \subset I \smallsetminus E$ be any closed interval with nonempty interior. Let $g_1 \in \pi^{-1}(x)$. We consider the discussion of §4 for J in place of I and $g_i = g_1$ for all i. Then there exist a reductive subgroup $H \in \mathscr{H}$ and $h_1 \in G$ and $g \in g_1\Gamma$ such that $AW \subset h_1 H h_1^{-1}$ and by (62),

$$u(\varphi(J)) \subset N \cap (N^- M h_1 H g^{-1}). \tag{71}$$

Without loss of generality we may assume that H is a smallest dimensional subgroup of $\mathscr{H}$ such that (71) holds. Also, since φ is an analytic function, we have

$$u(\varphi(I)) \subset N \cap (N^- M h_1 H g^{-1}). \tag{72}$$

Therefore there exists an analytic map $\zeta : I \to M$ such that,

$$u(\varphi(t)) \in N^- \zeta(t) h_1 H g^{-1}, \quad \forall t \in I,$$

and hence

$$\lim_{\substack{\alpha(a)\to\infty \\ a\in A^+}} a z(t) u(\varphi(t)) g \pi(H) = z(t)\zeta(t) h_1 \pi(H), \quad \forall t \in I. \tag{73}$$

Let $\lambda_i|_J$ denote the probability measure as defined in (29) for J in place of I. If there exists a sequence $j_k \stackrel{k\to\infty}{\longrightarrow} \infty$ such that $\lambda_{j_k}|_J \stackrel{k\to\infty}{\longrightarrow} \lambda'$, then by (73) we have

$$\operatorname{supp}(\lambda') \subset \{z(t)\zeta(t) : t \in J\} \cdot h_1 \pi(H) \subset \pi(N(H,W)). \tag{74}$$

Moreover by Theorem 3.1, λ' is W-invariant. According to the discussion as in §4, we deduce that $\lambda'(\pi(S(H,W)) = 0$, because otherwise (71) would hold for a strictly smaller dimensional subgroup in place of H and some $g \in \pi^{-1}(x)$. Therefore, by Proposition 4.8 and (74), for any $f \in \mathrm{C_c}(G/\Gamma)$,

$$\int_{G/\Gamma} f \,\mathrm{d}\lambda' = \frac{1}{|J|} \int_{t\in J} \left(\int_{y\in\pi(H)} f(z(t)\zeta(t)h_1 y) \,\mathrm{d}\,\mu_H(y) \right) \mathrm{d}t.$$

In particular, the right hand side is independent of the choice of the subsequence $\{j_k\}_{k\in\mathbb{N}}$. Therefore due to Corollary 2.5, we conclude that for any $f \in \mathrm{C_c}(G/\Gamma)$,

$$\begin{aligned} &\lim_{\substack{\alpha(a)\to\infty \\ a\in A^+}} \textstyle\int_{t\in J} f(a z(t) u(\varphi(t)) \pi(g)) \,\mathrm{d}t \\ &\qquad = \textstyle\int_{t\in J} \left(\int_{y\in\pi(H)} f(z(t)\zeta(t)h_1 y) \,\mathrm{d}\mu_H(y) \right) \mathrm{d}t. \end{aligned} \tag{75}$$

Let $\lambda_t = z_0(t)\zeta(t)\mu_H = z(t)\zeta(t)$ for all $t \in I$, where $z_0(t) \in M$ such that $z(t) \in z_0(t)A$. We apply Lemma 5.1 for $\theta_i(t) = z(t)u(\varphi(t))\pi(g)$,

$w_i(t) = z(t)^{-1}$, and $z_i(t) = e$ for all $t \in I$ and $i \in \mathbb{N}$. Then (13) follows from (75). □

Proof of Theorem 1.7. Let a sequence $g_i \to g$ in G be such that $x_i = \pi(g_i)$. For J as in the first paragraph of the proof of Theorem 1.8, we consider the discussion of §4. By Proposition 4.8 exists a reductive subgroup $H \in \mathscr{H}$, $h_1 \in G$, and $\gamma \in \Gamma$ such that $AW \subset h_1 H h_1^{-1}$ and by (62)

$$u(\varphi(J)) \subset N \cap (P^- h_1 H (g\gamma)^{-1}).$$

Therefore by (7), (10), and the analyticity of φ,

$$S(\varphi(I)) = p(u(\varphi(I))) \subset p(h_1 H \gamma^{-1} g^{-1}) = p(H_1 h_2),$$

where H_1 is the noncompact simple factor of $h_1 H h_1^{-1}$ containing A and $h_2 = h_1 \gamma^{-1} g^{-1} \in G$. In fact, we can express $H_1 = k_1 \operatorname{SO}(m, 1) k_1^{-1}$ for some $k_1 \in M$ and $2 \le m \le n$. Therefore,

$$S(\varphi(I)) \subset p(\operatorname{SO}(m, 1) h_3), \tag{76}$$

where $h_3 = k_1^{-1} h_2$. Now $p(\operatorname{SO}(m, 1)) \cong \mathbb{S}^{m-1}$, and under the map $p : G \to \mathbb{S}^{n-1}$, the right action of h_3 on G corresponds to a conformal transformation on $\mathbb{S}^{n-1}$. Therefore by (76), if $H \neq G$ then $m < n$ and $S(\varphi(I))$ is contained in an m-dimensional affine subspace of $\mathbb{R}^{n-1}$ intersecting $\mathbb{S}^{n-1}$. Since $\varphi(I)$ is not contained in an affine hyperplane or a sphere in $\mathbb{R}^{n-1}$ and S is the inverse of stereographic projection, $S(\varphi(I))$ is not contained in a proper subsphere of $\mathbb{S}^{n-1}$. Therefore we conclude that $H = G$.

For each $i \in \mathbb{N}$, we define $\lambda_i|_J$ as in (29) for J in place of I. Now if $j_k \to \infty$ is any sequence in $\mathbb{N}$ such that $\lambda_{j_k}|_J \xrightarrow{k\to\infty} \lambda_J$ in the space of probability measures, then by the discussion as in §4, by our choice of H as in Theorem 4.1, and since $H = G$, we have $\lambda_J(\pi(S(G, W)) = 0$. Hence almost all W-ergodic components of λ_J are G-invariant. Thus $\lambda_J = \mu_G$. Therefore by Corollary 2.5 we conclude that $\lambda_i|_J \to \mu_G$ as $i \to \infty$. Now the conclusion of theorem follows from (70) of Lemma 5.1. □

Proof of Theorem 1.6. Suppose that (11) fails to hold for a sequence of positive reals $R_i \to \infty$. Then there exists a sequence $\{a_i\} \subset A^+$ such that $\alpha(a_i) \ge R_i$ and a sequence $\{x_i\} \subset \mathcal{K}$ such that

$$\left| \frac{1}{|I|} \int_I f(a_i \theta(t) x_i))\, \mathrm{d}t - \int_{G/\Gamma} f \, \mathrm{d}\lambda_G \right| > \epsilon, \qquad \forall i \in \mathbb{N}. \tag{77}$$

Since $\mathcal{K}$ is compact, by passing to a subsequence, we may assume that $x_i \to x$ in G/Γ.

By Bruhat decomposition, $G = P^-N \cup P^-k_0$, where $k_0 \in K$ such that $k_0ak_0^{-1} = a^{-1}$ for all $a \in A$. Also the map $P^- \times N \to G \smallsetminus \{P^-k_0\}$ given by $P^- \times N \ni (b,u) \mapsto bu$ is an invertible analytic map. Since $p(\theta(I))$ is not a singleton set, and θ is analytic, the set $\{t \in I : \theta(t) \in P^-k_0\}$ is finite. As noted earlier, it is enough to prove the result for all closed subintervals of I with nonempty interiors and not intersecting this finite set. Hence without loss of generality we may assume that $\theta(t) \notin P^-k_0$ for all $t \in I$. Thus we obtain analytic maps $\varphi : I \to \mathbb{R}^{n-1}$ and $\zeta : I \to P^-$ such that

$$\theta(t) = \zeta(t)u(\varphi(t)), \quad \forall t \in I. \tag{78}$$

Then by (7) and (10), $S(\varphi(I)) = p(\theta(I))$. By our assumption, $p(\theta(I))$ is not contained in any hyperplane of $\mathbb{R}^n$ intersecting $\mathbb{S}^{n-1}$. Therefore, since S is the inverse of stereographic projection, $\varphi(I)$ is not contained in any hyperplane or a sphere of $\mathbb{R}^{n-1}$. Therefore by Theorem 1.7, for any subinterval $J \subset I$ with nonempty interior,

$$\lim_{i\to\infty} \frac{1}{|J|} \int_J f(a_i u(\varphi(t))x_i)\,\mathrm{d}t = \int_{G/\Gamma} f\,\mathrm{d}\mu_G, \tag{79}$$

Now by Lemma 5.1,

$$\lim_{i\to\infty} \frac{1}{|I|} \int_I f(a_i \zeta(t) u(\varphi(t))x_i)\,\mathrm{d}t = \int_{G/\Gamma} f\,\mathrm{d}\mu_G. \tag{80}$$

Now (78) and (80) contradict (77). □

Proof of Theorem 1.2. Let $G = \mathrm{SO}(n,1)$, $K = \mathrm{SO}(n)$, and let P^- be a maximal parabolic subgroup of G such that $P^- \cap K = \mathrm{SO}(n-1)$. Let A the maximal $\mathbb{R}$-diagonalizable subgroup of G centralizing $P^- \cap K$. Then $A \subset P^-$. Now G admits a transitive right action on $\mathrm{T}^1(\mathbb{H}^n)$ via isometries. We fix $\tilde{x}_0 \in \mathbb{H}^n$ such that $K = \mathrm{Stab}_G(\tilde{x}_0)$, and we fix $v_0 \in S_{\tilde{x}_0}(\mathbb{H}^n)$ such that

$$\mathrm{K}_0 := \mathrm{Stab}_K(v_0) = \mathrm{Z}_\mathrm{G}(A) \cap K = \mathrm{SO}(n-1).$$

Thus $\mathrm{T}^1(\mathbb{H}^n) \cong \mathrm{K}_0 \backslash G$ and $S_{\tilde{x}_0}(\mathbb{H}^n) \cong \mathrm{K}_0 \backslash K$. The under this isomorphism, the geodesic flow $\{\tilde{g}_t\}$ on $\mathrm{T}^1(\mathbb{H}^n)$ corresponds to the action of $\{a_t\} = A$ on $\mathrm{K}_0 \backslash G$ by left multiplications, where $\alpha(a_t) = e^{\tau t}$ for all $t \in \mathbb{R}$ and some $\tau > 0$.

There exists a discrete subgroup Γ of G such that $\pi : \mathbb{H}^n \to M$ factors through $\mathbb{H}^n/\Gamma$ and $M \cong \mathbb{H}^n/\Gamma$ as isometric Riemannian manifolds. Hence $\mathrm{T}^1(M) \cong \mathrm{K}_0 \backslash G/\Gamma$, and the geodesic flow $\{g_t\}$ on $\mathrm{T}^1(M)$ corresponds to the left action of $\{a_t\}$ on $\mathrm{K}_0 \backslash G/\Gamma$.

There exists an analytic map $\theta : I \to G$ such that

$$\psi(t) = \mathrm{D}\,\pi(v_0\theta(t)), \quad \forall t \in I.$$

As in the proof of Theorem 1.6, let $\varphi : I \to \mathbb{R}^{n-1}$ be the map such that (78) holds; that is, $\theta(t) \subset P^- u(\varphi(t))$ for all $t \in I$. Then by Theorem 1.8 for $x = e\Gamma \in G/\Gamma$, there exist $H \in \mathscr{H}$, $h_1 \in G$, and $\gamma \in \Gamma$ such that $Ah_1 \subset h_1 H$ and by (14), $u(\varphi(I)) \subset P^- h_1 H \gamma^{-1}$. Therefore

$$\theta(t) \subset P^- u(\varphi(t)) \subset P^- h_1 \gamma^{-1} = K_0 N^- h_1 H \gamma^{-1}, \quad \forall t \in I. \tag{81}$$

Therefore, since $\pi(v_0 K_0 g\Gamma) = \pi(v_0 g)$, we have

$$\pi(v_0 \theta(t)) \subset \pi(v_0 N^- h_1 H), \quad A \subset h_1 H h_1^{-1} \quad \text{and} \quad N \cap h_1 H h_1^{-1} \neq \{e\}.$$

Therefore there exists $k_1 \in \mathrm{K}_0$ such that

$$\mathrm{K}_0\, k_1 h_1 H h_1^{-1} k_1^{-1} = \mathrm{K}_0\, \mathrm{SO}(m,1), \quad \text{where } 2 \leq m \leq n.$$

Now $v_0\, \mathrm{SO}(m,1) \cong \mathrm{T}^1(\mathbb{H}^m)$, where $\mathbb{H}^m$ is isometrically embedded in $\mathbb{H}^n$. Since $H\Gamma/\Gamma$ is a closed subset of G/Γ,

$$\pi(v_0 h_1 H) = \pi(v_0\, \mathrm{SO}(m,1) k_1 h_1) = \pi(\mathrm{T}^1(\mathbb{H}^m) h_2),$$

is a closed subset of M, where $h_2 = k_1 h_1 \in G$. Therefore $\mathrm{K}_0\, h_1 H/(H \cap \Gamma)$ corresponds the embedding of $\mathrm{D}\,\Phi(\mathrm{T}^1(M_1))$ in $\mathrm{T}^1(M)$ which is the derivative of a totally geodesic immersion Φ of a hyperbolic manifold M_1 in M (see [13, §2] for the details). It may also be noted that the projection of $h_1 H h_1^{-1}$-invariant probability measure, say μ_1, on $h_1 H/(H \cap \Gamma)$ onto $\mathrm{K}_0 \backslash G/\Gamma \cong M$, say $\bar{\mu}_1$, is same as the projection under $\mathrm{D}\,\Phi$ of the normalized measure on $\mathrm{T}^1(M_1)$ associated to the Riemannian volume form on M_1.

By (13) of Theorem 1.8, for any subinterval J of I with nonempty interior and any $f \in \mathrm{C_c}(\mathrm{K}_0 \backslash G/\Gamma)$, we have

$$\lim_{t \to \infty} \frac{1}{|J|} \int_J f(\mathrm{K}_0\, a_t u(\varphi(s))\Gamma)\, \mathrm{d}s = \int_{\mathrm{K}_0\, h_1 H\Gamma/\Gamma} f(y)\, \mathrm{d}\bar{\mu}_1(y). \tag{82}$$

Recall that $\theta(s) \in P^- u(\varphi(s))$ for all $s \in I$, and $\bar{\mu}_1$ is $\mathrm{Z_G}(A)$-invariant with respect to the left action. Therefore by Lemma 5.1,

$$\lim_{t \to \infty} \frac{1}{|I|} \int_I f(\mathrm{K}_0\, a_t \theta(s)\Gamma)\, \mathrm{d}s = \int_{\mathrm{K}_0\, h_1 H\Gamma/\Gamma} f(y)\, \mathrm{d}\bar{\mu}_1(y). \tag{83}$$

Now in view of the relation between the closed $h_1 H h_1^{-1}$-orbits with totally geodesic immersions of finite volume hyperbolic manifolds as described above, (83) implies (3). □

Proof of Corollary 1.5. Let $\tilde{x} \in \mathbb{H}^n$ such that $x = \pi(\tilde{x})$. We can identify $\mathrm{T}^1{}_x(M)$, the unit tangent sphere at x, with $\mathrm{T}^1{}_{\tilde{x}}(\mathbb{H}^n)$, which in turn identifies with the ideal boundary sphere $\partial\mathbb{H}^n$ via the visual map. Since all these identifications are conformal, we conclude that $\mathrm{Vis}(\tilde{\theta}(I))$ is not contained in any proper subsphere of $\partial\mathbb{H}^n$. Therefore in terms of the notation in Remark 1.1, $\mathbb{S}^{k-1} = \partial\mathbb{H}^n$, and we conclude

that $M_1 = M$ and Φ is the identity map. Now the conclusion follows from Theorem 1.6. $\square$

Proof of Theorem 1.4. The proof is similar to the proof of Corollary 1.5. $\square$

Proof of Theorem 1.1. We identify $\mathbb{S}^{n-1}$ with a hyperbolic sphere of radius 1 centered at 0 in $\mathbb{H}^n$ (in the unit Ball B^n-model), say S, and treat $\bar{\psi}$ as a map from I to S. For any $s \in I$, let $v_s \in \mathrm{T}^1{}_{\bar{\psi}(s)}(\mathbb{H}^n)$ be the unit vector normal to S which is also a tangent to the directed geodesic from 0 to $\bar{\psi}(s)$. We define an analytic curve $\psi : I \to \mathrm{T}^1(M)$ by

$$\psi(s) = (\pi(\bar{\psi}(s)), \mathrm{D}\,\pi(v_s)), \quad \forall s \in I.$$

Therefore the condition of Theorem 1.4 is satisfied, because

$$\mathrm{Vis}(\tilde{\psi}(s)) = \mathrm{Vis}((\bar{\psi}(s), v_s)) = \bar{\psi}(s), \forall s \in I,$$

and hence $\mathrm{Vis}(\tilde{\psi}(s))$ is not contained in a proper subsphere of $\partial\mathbb{H}^n$.

For any $\alpha > 0$, $\pi(\alpha\bar{\psi}(s)) = g_{t(\alpha)}\pi(\psi(s))$ for some $t(\alpha) > 0$ such that $t(\alpha) \to \infty$ as $\alpha \to 1^-$. Therefore (2) follows from Theorem 1.4. $\square$

5.1. **A stronger version.**

Theorem 5.2. *Let $G = \mathrm{SO}(n,1)$ and Γ a lattice in G. Let $\theta : I = [a,b] \to G$, where $a < b$, be an analytic map such that for any minimal parabolic subgroup P^- of G, the image of $\psi(I)$ in $P^-\backslash G \cong \mathbb{S}^{n-1}$ is not contained in any proper subsphere of $\mathbb{S}^{n-1}$. Then given any $f \in \mathrm{C_c}(G/\Gamma)$, a compact set $\mathcal{K} \subset G$ and an $\epsilon > 0$, there exists a compact set $\mathcal{C} \subset G$ such that*

$$\left| \frac{1}{|I|} \int_I f(g\theta(t)x)\,\mathrm{d}t - \int_{G/\Gamma} f\,\mathrm{d}\lambda_G \right| < \epsilon, \quad \forall x \in \mathcal{K} \text{ and } \forall g \in G \smallsetminus \mathcal{C}.$$

A proof of the above generalization of Theorem 1.6 can be given by similar arguments. The analogue of §4.3 is a little more delicate in this case. We do not include the proof here in order to have simpler proofs for all other results.

6. Scope for generalization and applications

The results of this article lead to obvious similar questions about expanding translates of (C,α)-good curves on horospherical subgroups of general semisimple Lie groups. Especially the affirmative answer to the following question has interesting applications to problems in Diophantine approximation [6, 7]:

Question 6.1. *Let $G = \mathrm{SL}(n+1,\mathbb{R})$, $\Gamma = \mathrm{SL}(n+1,\mathbb{Z})$, and μ_G denote the G-invariant probability measure on G/Γ. Let*

$$u(v) = \left(\begin{smallmatrix} 1 & v \\ 0 & I_n \end{smallmatrix}\right), \ \forall v \in \mathbb{R}^n, \ \text{and } a(t) = \mathrm{diag}(e^{nt}, e^{-t}, \dots, e^{-t}), \ \forall t \in \mathbb{R}.$$

Let $\varphi : [0,1] \to \mathbb{R}^n$ be an analytic (or a (C,α)-good) curve such that its image is not contained in any proper affine hyperplane in $\mathbb{R}^n$. Then is it true that for any $x \in G/\Gamma$ and any $f \in \mathrm{C_c}(G/\Gamma)$,

$$\lim_{t\to\infty} \int_0^1 f(a(t)u(\varphi(s))x)\,\mathrm{d}s = \int_{G/\Gamma} f\,\mathrm{d}\mu_G\,? \tag{84}$$

The main result of [6] provides a very good estimate on the rate of nondivergence of this translated measure. The method of this article is applicable to show that, after a suitable modification of the curve by elements form the centralizer of $\{a(t)\}$, the limiting measure is invariant under a unipotent one-parameter subgroup of the form $\{u(sw_0)\}$ for some $w_0 \in \mathbb{R}^n \setminus \{0\}$. Also the method to study behaviour of expanded trajectories near the singular sets is applicable here. Obtaining an analogue of Lemma 2.3 in order to derive algebraic consequences of Proposition 4.5 is the main difficulty in this problem.

Since the initial submission of this article, the author [15] has answered Question 6.1 in affirmation for analytic curves.

In another direction, it is still an open question to prove the exact analogue of Theorem 1.4 for the actions of $\mathrm{SO}(n,1)$ on homogeneous spaces of larger Lie group G containing $\mathrm{SO}(n,1)$; see [5].

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